Numerical modeling of tornado-like atmospheric vortices:
Model development, validation, and limitations

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I. Introduction

Numerical models are necessary to supplement observational datasets of atmospheric phenomena, especially those which are difficult to effectively sample due to spatial, temporal, technological, and other similar limitations. Appropriate simulations of a phenomenon enable researchers to fill in gaps of knowledge and support data collected in the field or laboratory; models themselves, however, are subject to the same aforementioned limitations that restrict observational datasets. Given the current state of knowledge and computing capabilities, we seek to develop or identify an atmospheric numerical model capable of properly representing a phenomenon that is difficult to effectively sample: the tornado[1].

Simulations of tornadoes have evolved significantly over the past half century. Wilhelmson and Wicker (2001) provide a thorough review of this progression from two-dimensional axisymmetric models to three-dimensional axisymmetric or asymmetric representations. Two-dimensional axisymmetric simulations of tornadoes allowed researchers to focus on flow near and around a vortex core with minimal computational resources. Howells and coauthors (1988) provided a detailed study of axisymmetric numerical models in order to identify sensitivities in the simulations. Although technological resources now allow for more sophisticated model set-up (i.e. three-dimensional asymmetry), the utility of two-dimensional axisymmetric vortex modeling persists even today (e.g. Wood and Brown 1997). Ward's (1972) vortex chamber, although not a numerical model, provided an outlet for some of the initial studies of factors influencing vortex structure and dynamics; later, as computing power increased, Rotunno (1984) was able to use a three-dimensional numerical model to recreate the laboratory chamber simulations. With this and many other asymmetric simulations since the

1980's (e.g. Fiedler 1998), features of multiple vortex tornadoes, such as Fujita's (1971) descriptively-named "suction vortices," could be better reproduced. It is important to note, however, that axisymmetric vortex models are also still being successfully applied to multiple vortex tornadoes and other vortex phenomena (such as tropical cyclones) two decades later (e.g. Wurman and Lee 2005, Bryan and Rotunno 2009, respectively).

In time, three-dimensional vortices began to be modeled using large-eddy simulations (LES) to better represent turbulent processes (e.g. Lewellen 1993, Lewellen et al. 1997). The LES acts as the subgrid flux and the method of transport for energy between resolvable and unresolvable scales in cloud-resolving models, although the appropriateness of its usage for most research purposes may be limited to relatively small (O(100m)) horizontal grid spacing (Bryan et al. 2003). This horizontal grid spacing point hints at another conflicting problem in the modeling of tornadoes. Additional computing resources have allowed for greater domain coverage and hence increased representation of the parent storm of the tornado; however, as larger-scale processes that contribute to the genesis and sustainment of the vortex have been better-resolved, the resolution of the vortex itself has been correspondingly degraded (Wilhelmson and Wicker 2001).

In the spirit of the preceding brief history of tornado research models, the objectives of this paper are to:

1) identify appropriate governing equations and attendant assumptions for use in a numerical model simulating tornado-like atmospheric vortices, and attempt to validate the model despite the sampling limitations of the phenomenon,

2) discuss potential limitations to the model based on the assumptions implicit in our governing equations as well as other potential caveats as identified or hypothesized in the

published literature, and

3) outline future research directions based on the benefits and restrictions of our chosen model.

Each of the numbered points represents the content of sections II through IV, respectively; concluding remarks will be saved for section V. For the purposes of this research we are interested primarily in the representation of the vortex itself, not the vortex as associated with and influenced by the parent storm; the potential ramifications of isolating the vortex for simulation will be discussed in section III.

II. Model development and validation

a. Determining the governing equations

Tornadoes are relatively short-lived, spatially-limited phenomena, with life spans ranging from seconds to hours and vortex sizes ranging from a few meters to a few kilometers. While some atmospheric vortices of similar spatial and temporal scale may be properly simulated in conditions of dry convection (i.e. dust devils, see Rennó et al. 1998), the formation mechanisms and environments of both supercellular and non-supercellular tornadoes necessitate the use of moist convective processes in defining the governing equations of our desired model. Bannon (2002) provides a detailed theoretical basis for determining the appropriate governing equations for moist convection; we will use this as a starting point for our desired vortex model. The paper develops equations that better represent clouds and their associated processes by recognizing through the choice of variables and equations that a cloud is a complex system consisting of various temperatures, phases, and fall velocities. Differences amongst hydrometeors and between the hydrometeors and the attendant air itself are acknowledged explicitly, limiting the initial assumptions in the derivation of the governing equations.

The equations developed by Bannon (2002) conserve mass and energy and allow for hydrometeors to retain thermodynamic and dynamic characteristics independent of the surrounding air and water vapor. As will be explained in the validation portion of this section (II.c), retaining some terms that are often neglected may have a significant impact on model accuracy. Although the following equations consider only two phases of hydrometeors (liquid and ice), the author asserts that a range of hydrometeor classifications may be included. At the most basic level (with minimum complicating variables) in a compressible atmosphere, the equations[2] as defined in Bannon (2002) include (1) the equation of state for dry air and water vapor:

$$p_a = \rho_a RT \text{ and } e = \frac{r_v}{\varepsilon} p_a \qquad (1)$$

(2-5) Continuity equations:

$$\frac{D\rho_a}{Dt} = -\rho_a \nabla \cdot u_a \qquad (2)$$

$$\frac{Dr_v}{Dt} = -\dot{r}_{cond} - \dot{r}_{dep} - \dot{r}_{dif} \qquad (3)$$

$$\frac{Dr_l}{Dt} = \dot{r}_{cond} - \dot{r}_{freez} - \dot{r}_{liqfallout} \qquad (4)$$

$$\frac{Dr_i}{Dt} = \dot{r}_{dep} + \dot{r}_{freez} - \dot{r}_{icefallout} \qquad (5)$$

[Equation (2) considers dry air while the latter three equations altogether form continuity for moist air. Combining equations (3-5) physically implies that, when compared to dry air, water loss is due to water vapor diffusion in addition to fallout of ice and liquid water.]

(6-8) Momentum equations:

$$\frac{Du_l}{Dt_l} = -\frac{1}{\rho_{drop}}\nabla p + g - \frac{v_l}{\tau_{vl}} \qquad (6)$$

$$\frac{Du_i}{Dt_i} = -\frac{1}{\rho_{part}}\nabla p + g - \frac{v_i}{\tau_{vi}} \qquad (7)$$

$$(1+r_v)\frac{Du_a}{Dt} = -\frac{(1-\alpha_h)}{\rho_a}\nabla p + \frac{1}{\rho_a}\nabla \cdot \sigma + (1+r_v)g + \dot{u}_m \qquad (8)$$

Where (9):

$$\dot{u}_m = r_l \frac{v_l}{\tau_{vl}} + r_i \frac{v_i}{\tau_{vi}} - \dot{u}_{phase} \qquad (9)$$

[In (6-7), the viscous decay terms are functions of hydrometeor shape and the Reynolds number. Equation (8) is the combined momentum equation for moist air.]

And (10-12) energy equations:

$$\frac{Di_l}{Dt_l} = c_l \frac{DT_l}{Dt_l} = \frac{n_{drop} l_v (T_l) \dot{m}_{cond}}{\rho_a r_l} + \dot{q}_{radl} - c_l \frac{(T_l - T)}{\tau_{cl}} = \dot{q}_l \qquad (10)$$

$$\frac{Di_i}{Dt_i} = c_i \frac{DT_i}{Dt_i} = \frac{n_{part} l_s (T_i) \dot{m}_{dep}}{\rho_a r_i} + \dot{q}_{radi} - c_i \frac{(T_i - T)}{\tau_{ci}} = \dot{q}_i \qquad (11)$$

$$\rho_a c_{vm} \frac{DT}{D\bar{t}} = -p\nabla \cdot \bar{u} + \rho_a \dot{q}_m \qquad (12)$$

[Here, it is assumed that rates of change in the “m” terms are known from microphysics.]

Within these equations, it is assumed that the time scale of dynamical interest is large compared to the scale of heat and momentum transfer from the hydrometeors. Furthermore, the equations do not consider small-scale eddies which are relevant to atmospheric vortices. Appropriate parameterizations for these processes will be discussed in section II.d. For now, let us consider how to solve these equations numerically for incorporation into and validation of our vortex model.

b. Numerical solutions of the governing equations

Bryan and Fritsch (2002) utilize the foundations supplied by Bannon (2002) to develop a three-dimensional, non-hydrostatic, mass- and energy-conserving cloud model that considers perturbations from a hydrostatic base state. The numerical equations described hereafter act as the basis for the First-Generation Pennsylvania State University/National Center for Atmospheric Research Cloud Model, a numerical model currently active in the atmospheric community developed for use with various phenomena associated with moist convection (CM1; documentation and model available online at http://www.mmm.ucar.edu/people/bryan/cm1/; see also Bryan 2002). In addition to equations (1) through (12), governing equations for pressure and potential temeperature are necessary (provided in summation notation):

$$\frac{D\ln\pi}{Dt} = -\frac{R}{c_p}\frac{c_{pml}}{c_{vml}}\frac{du_j}{dx_j} - \frac{R}{c_p}\left(\frac{L_v}{c_{vml}T} - \frac{R_v c_{pml}}{R_m c_{vml}}\right)\frac{Dr_v}{Dt} \qquad (13)$$

$$\frac{D\ln\theta}{Dt} = -\left(\frac{R_m}{c_{vml}} - \frac{Rc_{pml}}{c_p c_{vml}}\right)\frac{du_j}{dx_j} - \left[\frac{c_v L_v}{c_{vml}c_p T} - \frac{R_v}{c_{vml}}\left(1 - \frac{R}{c_p}\frac{c_{pml}}{R_m}\right)\right]\frac{Dr_v}{Dt} \qquad (14)$$

Where the nondimensional pressure and potential temperature are defined as follows:

$$\pi = \left(\frac{p}{1000}\right)^{\frac{R}{c_p}} \qquad (15)$$

$$\theta = \frac{T}{\pi} \qquad (16)$$

Note that (13) and (14) are often seen simplified to ignore the differences in specific heats of various water phases. The authors argue (and later go on to show) that retention of these terms is necessary for conservation of energy, which is often neglected and considered unnecessary in other models. It is also acknowledged that due to the terms included, potential temperature is never truly conserved when water is present. They show, however, that using an alternate form is not only more computationally intensive, but produces minimally different results from the approximations implicit in using equations (15) and (16).

Numerical solutions to the governing equations (1) through (16) are as follows, in summation notation[3]:

$$\frac{du_i}{dt} = -\frac{d(u_i u_j)}{dx_j} + u_i \frac{du_j}{dx_j} - c_p \theta_p \frac{d\pi'}{dx_i} + \delta_{i3} g \left(\frac{\theta_p}{\theta_{p0}} - 1 \right) + K_d \frac{dD}{dx_i} \qquad (17)$$

$$\frac{d\pi}{dt} = -\frac{d(u_j \pi)}{dx_j} + \pi \frac{du_j}{dx_j} - \pi \frac{R}{c_p} \frac{c_{pml}}{c_{vml}} \frac{du_j}{dx_j} + \frac{R}{c_p} \left(\frac{L_v}{c_{vml}\theta} - \pi \frac{R_v c_{pml}}{c_p c_{vml}} \right) \dot{r}_{cond} \qquad (18)$$

$$\frac{d\theta}{dt} = -\frac{d(u_j \theta)}{dx_j} + \vartheta \frac{du_j}{dx_j} - \theta \left(\frac{R_m}{c_{vml}} - \frac{R c_{pml}}{c_p c_{vml}} \right) \frac{du_j}{dx_j} + \left[\frac{c_v L_v}{c_{vml} c_p \pi} - \theta \frac{R_v}{c_{vml}} \left(1 - \frac{R}{c_p} \frac{c_{pml}}{R_m} \right) \right] \dot{r}_{cond} \qquad (19)$$

$$\frac{dr_v}{dt} = -\frac{d(u_j r_v)}{dx_j} + r_v \frac{du_j}{dx_j} - \dot{r}_{cond} \qquad (20)$$

$$\frac{dr_c}{dt} = -\frac{d(u_j r_c)}{dx_j} + r_c \frac{du_j}{dx_j} - \dot{r}_{cond} \qquad (21)$$

In an effort to further increase conservation of variables in the model, the advection terms are not written in the traditional form but rather as the sum of a flux and divergence term (i.e. the first two terms on the right hand side of the previous equations). It is imperative to note here that, as is the case with nearly any numerical model, even with the additional measures included to help conserve mass and energy, neither is perfectly conserved due to truncation errors in running the model. The model attempts to account for change of phase with a saturation adjustment method in which the model is first advanced with a dynamical step followed by a microphysical step; the latter step includes any pressure perturbations effected by a phase change (Bryan and Fritsch 2002).

c. Model validation

Given the previously-laid theoretical groundwork for moist convection, the lingering question is: how do we validate these assumptions in a model framework? We have no appropriate method with which hydrometeor type, cloud thermodynamics, etc. can be accurately measured on correct spatial and temporal scales, so we need another way to benchmark our model. Oreskes et al. (1994) discuss the difficult nature of validation in earth systems models, bluntly opening their article by stating that “verification and validation of numerical models of natural systems is impossible.” While some may debate whether the paper is “bogged down” in an argument of semantics, their assertion that “verification” implies absolute truth while “validation” implies legitimacy is relevant here. We seek to validate the legitimacy of our model in the framework of a benchmark case.

Bryan and Fritsch (2002) offer the following approach to validating moist numerical models: given a specific base state and initial conditions, compare the general results and loss of mass and energy of a moist simulation with a dry simulation; if the differences are minimal, the moist simulation can be used as a benchmark against which other moist model simulations can be validated against. The authors describe other methods through which models are often validated, but argue that providing a benchmark, especially when considering moist convective processes, is most reliable. For the governing equations of the benchmark solution (described in section II.b), a compressible atmosphere that neglects anelastic processes (by integrating relevant equations on smaller time steps), fallout of hydrometeors, ice physics, Coriolis, and subgrid turbulence is assumed. Similar to the assumptions implicit in the Bannon (2002) equations, it is apparent from (at minimum) the final ignored term, subgrid turbulence, that additional processes and/or parameterizations will be necessary to include to properly resolve tornadic flow; for benchmarking purposes, however, these approximations appear reasonable.

The authors compare the results of their benchmark moist simulation of a rising thermal to that of a dry simulation and determine that (1) mass and energy are well-conserved and (2) only minimal differences exist between the solutions of the runs themselves. It is noted that in the benchmark solution, no term in the preceding governing equations is ignored for computational simplicity. The authors go on to test four additional model formulations against their dry simulation to determine if any other set-up in the literature could be used as a benchmark. The formulations use varying degrees of approximations that are not present in the original benchmark simulation; in the absence of water, all the formulations would be the same. These approximated simulations do not agree well with the benchmark solution, further suggesting that conservation of thermodynamic variables is more important than perhaps previously anticipated. Limitations of the applicability of these results and the CM1 model itself will be discussed further in section III.

d. Resolution requirements

It is important to discuss the spatial and temporal scales in which our previously-discussed model is most relevant for our desired phenomenon—tornadoes. Fortunately, Bryan and coauthors (2003) consider this question with respect to the aforementioned model and the inclusion of turbulence, a neglected process for the purpose of the benchmark solution. The authors show that, in order for a numerical model to resolve all scales of turbulence, 0.1 mm horizontal grid spacing would be necessary. Because this is not computationally feasible, we can effectively remove the smallest scales from a flow by filtering the Navier-Stokes equations. Upon filtering the equations, we obtain an unknown term that must be parameterized; for this common cloud model turbulence closure problem, we use a large-eddy simulation (LES) to handle fluxes on the subgrid scale as well as transfer of energy between the resolved and unresolved scales (Bryan et al. 2003).

In order to determine the appropriate grid spacing, the authors suggest that the scale of the phenomenon to be simulated (*l*) must be much larger than the grid spacing (Δ). How many orders of magnitude qualify as "much larger?" This question is addressed in the context of the Reynolds number and some planetary boundary layer (PBL) turbulence studies. The relationship between the Reynolds number and our other two terms is defined as follows:

$$R \sim \left(\frac{l}{\Delta}\right)^{4/3} \qquad (22)$$

This relationship confirms the aforementioned assertion that in order to resolve turbulent flow (R > 1) the scale of the phenomenon being sampled must be much larger than the grid spacing. The authors continue this analysis with respect to the PBL studies, in which a relationship of $l/\Delta \sim 100$ was sufficient to resolve turbulence. In this case, a 10 km phenomenon necessitates a grid spacing of 100 m to properly resolve turbulent processes. The authors continue to test this hypothesis and find that the higher-resolution simulations better recreate squall line features than simulations conducted at 1 km grid spacing (Bryan et al. 2003). Adlerman and Droegemeier (2002) find similar resolution dependency of cyclic mesocyclogenesis in supercells; by decreasing their horizontal grid spacing, their simulations evolved from a steady to a cycling supercell.

Based on the preceding analysis, for our purposes in simulating tornado-like vortices (*O*(100m) on average, Davies-Jones et al. 2001) we would require grid spacing of 1 m to properly resolve the subgrid-scale turbulent processes. This is not a particularly feasible computational solution. Preliminary work shows that others have successfully used CM1 to simulate the development of a tornado cyclone (not the vortex itself) with 100 m horizontal grid spacing and 1 s large time steps and 0.1 s small time steps (Markowski et al. 2010). The duration of the time steps is desirable for our purposes, given that multiple modeling and observational studies have shown that tornado vortex structure may change rapidly over very small time scales (e.g. Wurman 2002). It is probable that for the purposes of our research similar sensitivity tests will need to be performed to determine the appropriate horizontal grid spacing for our vortex simulations. At minimum we should begin with a resolution of 100 m and work our way down to smaller scales, if and when it is computationally plausible. Future research directions will be discussed in section IV.

III. Potential limitations and additional caveats

As previously mentioned, some of the assumptions that went into the benchmark solution were described by the authors themselves as "unphysical" (Bryan and Fritsch 2002). Although this is true, it is necessary in order to construct a situation in which the solution is accurately known. The primary purpose of the benchmark simulation is to test the validity of the model's governing equations; CM1 successfully does that by retaining all the terms in its governing equations to allow for conservation of mass and energy. This benchmark simulation is likely the best of its kind with respect to moist convection at this time.

Fiedler (1995) argues against modeling tornadoes as entities separate from their associated storm. He first acknowledges that, as mentioned in Wilhelmson and Wicker (2001), it is tempting to focus on the vortex core itself to avoid resolution degradation by incorporating the parent storm. The problem with "uncoupling" the tornado from its parent, however, is that the the latter directs and controls the former. With respect to our research, it is first important to note that, at this point in time, we are not interested in tornadogenesis, only the effect/structure/velocity of a vortex once it is present. Furthermore, Fiedler's study considers only subcritical vortices and acknowledges that a supercritical vortex may be less influenced by the same interactions as the former, but the process of vortex breakdown may not occur without outside factors. For our research, we will be largely interested in supercritical vortices, but also potentially the process of vortex breakdown itself. It is apparent from Fiedler's arguments that, in addition to our sensitivity tests for horizontal grid spacing (see section II.d), we may have to perform experiments with and without the parent storm to determine its impact in the context of CM1 modeling. It is potentially relevant to note here that, despite the aforementioned arguments, Fiedler also went on to model isolated, three-dimensional, asymmetric tornadoes with suction vortices due to the focused nature of the studies (e.g. Fiedler 1998).

IV. Anticipated research directions

Given the preceding discussion, we now seek to briefly outline a future research plan:

- Determine appropriate initial conditions, grid spacing, and time steps for simulating a single tornadic vortex.
- Perform sensitivity tests of grid spacing and time steps. This may require developing a benchmark vortex simulation for testing.

- If necessary, repeat simulations with parent storm included; note that this will necessarily degrade vortex resolution and otherwise alter the results of the first two bullets based on the increase in necessary grid spacing outlined in section II.d.
- Repeat the first two bullets for multiple vortex events; adjust based on results of the third bullet point.
- Sample all the simulated vortices with simulated Doppler radars (outside the scope of this numerical modeling-focused paper).

V. Conclusions

Relevant governing equations for simulations of moist convection have been derived, applied numerically, and validated. Appropriate turbulence closures, horizontal grid spacing, and time steps have been discussed with respect to tornado-like atmospheric vortices. Despite the noted limitations of our chosen model, CM1, we have shown it to be the best available model for our research purposes, as outlined in section IV. Literature by Bryan, Fiedler, and others has encouraged us to develop a benchmark solution of our own and conduct sensitivity tests to determine the most appropriate model set-up for our chosen phenomenon; this is especially pertinent since literature is scarce on the use of CM1 to model tornado-like vortices in isolation from the attendant storm.

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Appendix A: Footnotes

[1] Defined here by the official AMS Glossary entry for "tornado: a violently rotating column of air, in contact with the surface, pendant from a cumuliform cloud, and often (but not always) visible as a funnel cloud" (available at http://amsglossary.allenpress.com).

[2] All equation terms are defined in Appendix B.

[3] Equations (17) through (21) should have the d replaced with ∂

Appendix B: Equations

(In order of appearance as defined or specified in Bannon (2002) and Bryan and Fritsch (2002))

p_a Pressure of dry air

ρ_a Density of dry air

R Ideal gas constant for dry air

T Temperature of dry air and water vapor

e Pressure of water vapor

r_v Water vapor mixing ratio

ε Ratio of molecular weight of water to air

u_a Velocity of dry air

$\dot{r}_{cond}$ Rate per unit mass of dry air of condensation

$\dot{r}_{dep}$ Rate per unit mass of dry air of deposition

$\dot{r}_{dif}$ Rate of diffusion of water vapor per unit mass of dry air

r_l Liquid water mixing ratio

$\dot{r}_{freez}$ Rate per unit mass of dry air of freezing

$\dot{r}_{fallout}$ Fallout rate of water per unit mass of dry air

u_l Velocity of liquid drop

ρ_{drop} Density of liquid drop

τ_{vl} Viscous timescale for liquid drop

g Gravitational acceleration

p Total pressure

u_i Velocity of ice particle

τ_{vi} Viscous timescale for ice particle

α_h Volume fraction taken by hydrometeors

σ Moist air viscous stress tensor

$\dot{u}_{phase}$ Phase change rate of momentum loss per unit mass dry air

i_l Specific internal energy of liquid water

c_l Specific heat of liquid water

n_{drop} Number density of liquid drops

l_v Enthalpy of vaporizations

m_{cond} Mass rate of condensation

$\dot{q}_{radl}$ Heating rate of liquid drops per unit mass due to radiation

$\dot{q}_l$ Heating rate of liquid drops

n_{part} Number density of ice particles

m_{dep} Mass rate of deposition

c_{vm} Specific heat at constant volume of moist air

$\bar{u}$	Mass-weighted velocity of moist air
$\dot{q}_m$	Heating rate of moist air
R	Gas constant for dry air
R_m	Gas constant of moist air
R_v	Gas constant of water vapor
c_p	Specific heat of dry air (constant pressure)
c_v	Specific heat of dry air (constant volume)
c_{pml}	Specific heat of moist air (constant pressure)
c_{vml}	Specific heat of moist air (constant volume)
L_v	Latent heat of vaporization
θ_p	Density potential temperature
θ_{p0}	Initial density potential temperature
K_d	Divergence damper constant
D	$= \frac{\partial u_j}{\partial x_j}$